√SmarterMaths

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2019.

The 2019 paper for the HSC Mathematics course is its swansong after four long decades where the syllabus has remained intact.

This is our "Final Stretch HSC Revision Series" that we recommend motivated students aiming for a Band 5 or 6 result should ***attempt, carefully review and annotate*** to create a concise and high quality revision resource that they can refer back to.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create an extremely important revision set that ensures students cover a wide cross-section of the key areas we have carefully identified.

IMPORTANT: If students have been exposed to some of the questions in these worksheets during the year, all the better. Do not underestimate the crucial skill of developing speed through the exam and few revision practices are more effective in achieving this than revisiting quality questions for a second (and third) time.

HSC Final Study: EXT1 Topics 9-10 (~12% historical contribution)

Key Areas addressed by this worksheet

Topic 9: The Parabola

- Super consistent contributor – numerous examples of multi-part "standard" questions containing a range of difficulty levels are reviewed. Chunky mark allocations and historically low mean marks make this topic revision crucial in achieving high band results!
- Locus has proven very challenging and is an important revision focus in 2019 (not tested for the last 3 years);
- Generation of both tangent and normal equation proofs .. students should be able to do this – along with chord proof – *very quickly*. Not asked since 2011 but these proofs were common in the past.

Topic 10: Geometrical Applications of Calculus

- Curve sketching is a revision focus area as it has been under-examined in the last 5 years;
- The most common question type involving curves with asymptotes (vertical and horizontal) are reviewed;
- Max/min problems may also appear as a 2-3 mark part of a larger cross-topic question, as they did between 2012-2014 (this *does not include* any max/min elements of projectile or other Topic 14 questions). Note that this type of max/min problem (i.e. *ex-Topic 14)* has been absent in the last 4 exams and is covered in this revision set.

EXTENSION 1

2019 HSC Final Stretch Revision Series

- T9. Quadratics and The Parabola

- T10. Geometrical Applications of Calculus

Teacher: Smarter Maths

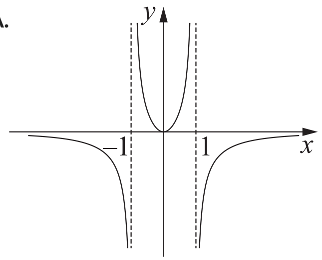
Exam Equivalent Time: 45 minutes *(based on HSC allocation of 1.5 minutes approx. per mark)*

Questions

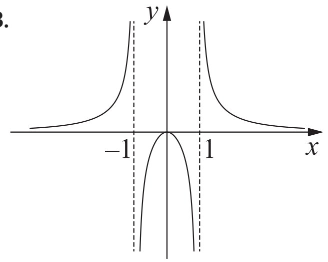
1. Geometry and Calculus, EXT1 2017 HSC 5 MC

Which graph best represents the function $y = \dfrac{2x^2}{1 - x^2}$?

A.



B.



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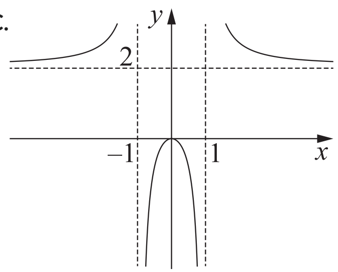
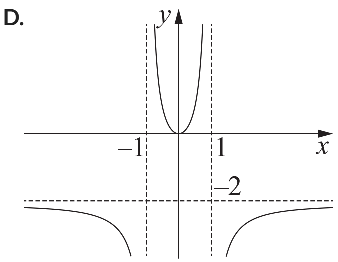
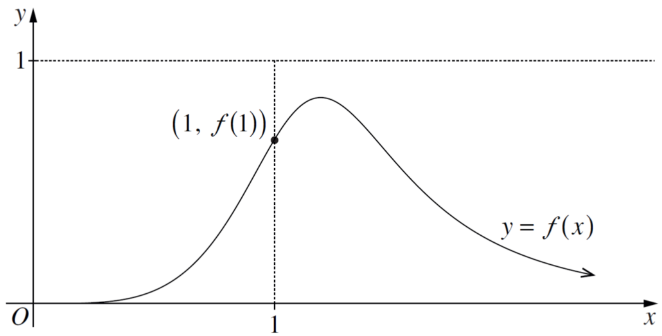


D.



2. Geometry and Calculus, EXT1 2016 HSC 9 MC

The diagram shows the graph of $y = f(x)$.



Which of the following is a correct statement?

(A) $f''(1) < f(1) < 1 < f'(1)$

(B) $f''(1) < f'(1) < f(1) < 1$

(C) $f(1) < 1 < f'(1) < f''(1)$

(D) $f'(1) < f(1) < 1 < f''(1)$

3. Geometry and Calculus, EXT1 2012 HSC 13b

(i) Find the horizontal asymptote of the graph

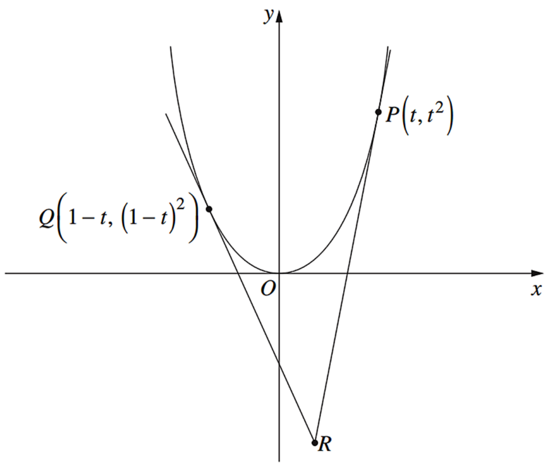
$$y = \frac{2x^2}{x^2 + 9}.$$ *(1 mark)*

(ii) Without the use of calculus, sketch the graph

$y = \dfrac{2x^2}{x^2 + 9}$, showing the asymptote found in part (i). *(2 marks)*

4. Quadratic, EXT1 2011 HSC 3b

The diagram shows two distinct points $P(t, t^2)$ and $Q(1 - t, (1 - t)^2)$ on the parabola $y = x^2$. The point R is the intersection of the tangents to the parabola at P and Q.



(i) Show that the equation of the tangent to the parabola at P is $y = 2tx - t^2$. *(2 marks)*

(ii) Using part (i), write down the equation of the tangent to the parabola at Q. *(1 mark)*

(iii) Show that the tangents at P and Q intersect at $R\left(\frac{1}{2}, t - t^2\right)$. *(2 marks)*

(iv) Describe the locus of R as t varies, stating any restriction on the y-coordinate. *(2 marks)*

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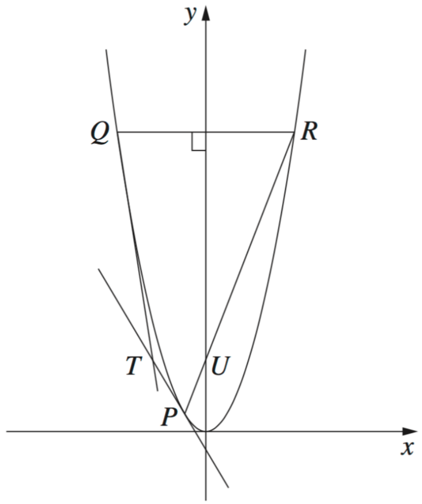
5. Quadratic, EXT1 2006 HSC 2c



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$ lie on the parabola $x^2 = 4ay$. The chord QR is perpendicular to the axis of the parabola. The chord PR meets the axis of the parabola at U.

The equation of the chord PR is $y = \frac{1}{2}(p + r)x - apr$. (Do NOT prove this.)

The equation of the tangent at P is $y = px - ap^2$. (Do NOT prove this.)

(i) Find the coordinates of U. *(1 mark)*

(ii) The tangents at P and Q meet at the point T. Show that the coordinates of T are $(a(p + q), apq)$. *(2 marks)*

(iii) Show that TU is perpendicular to the axis of the parabola. *(1 mark)*

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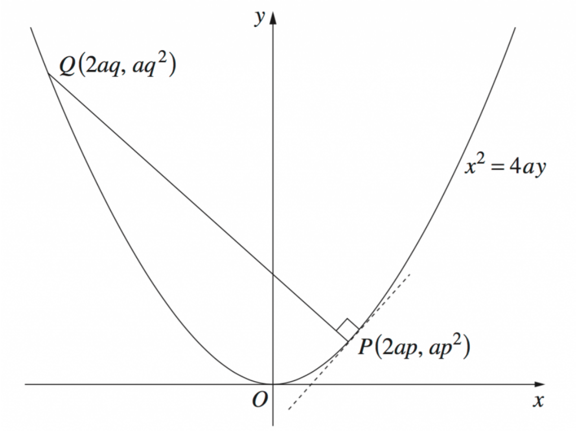
6. Quadratic, EXT1 2007 HSC 5d



The diagram shows a point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$. The normal to the parabola at P intersects the parabola again at $Q(2aq, aq^2)$.

The equation of PQ is $x + py - 2ap - ap^3 = 0$. (Do NOT prove this.)

(i) Prove that $p^2 + pq + 2 = 0$. *(1 mark)*

(ii) If the chords OP and OQ are perpendicular, show that $p^2 = 2$. *(2 marks)*

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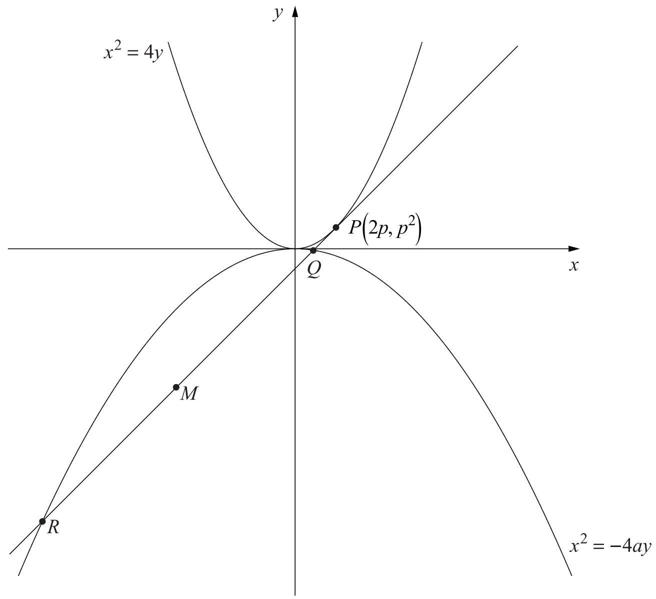
7. Quadratic, EXT1 2017 HSC 14b

Let $P(2p, p^2)$ be a point on the parabola $x^2 = 4y$.

The tangent to the parabola at P meets the parabola $x^2 = -4ay, a > 0$, at Q and R. Let M be the midpoint of QR.



(i) Show that the x coordinates of R and Q satisfy

$$x^2 + 4apx - 4ap^2 = 0.$$ *(2 marks)*

(ii) Show that the coordination of M are $(-2ap, -p^2(2a+1))$. *(2 marks)*

(iii) Find the value of a so that the point M always lies on the parabola $x^2 = -4y$. *(2 marks)*

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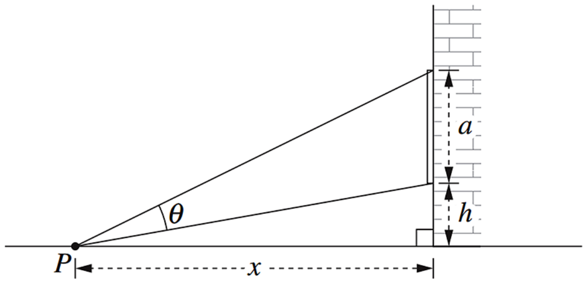
8. Geometry and Calculus, EXT1 2009 HSC 7b

A billboard of height a metres is mounted on the side of a building, with its bottom edge h metres above street level. The billboard subtends an angle θ at the point P, x metres from the building.



(i) Use the identity $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ to show that

$$\theta = \tan^{-1}\left[\frac{ax}{x^2 + h(a + h)}\right].$$ *(2 marks)*

(ii) The maximum value of θ occurs when $\dfrac{d\theta}{dx} = 0$ and x is positive.

Find the value of x for which θ is a maximum. *(3 marks)*

Worked Solutions

1. Geometry and Calculus, EXT1 2017 HSC 5 MC

$$\begin{aligned} y &= \frac{2x^2}{(1 - x^2)} \\ &= -\frac{(2 - 2x^2 - 2)}{(1 - x^2)} \\ &= -\frac{2(1 - x^2)}{(1 - x^2)} - \frac{2}{(1 - x^2)} \\ &= -2 - \frac{2}{(1 - x^2)} \end{aligned}$$

As $x \to \infty,\ y \to -2$

$\therefore$ Horizontal asymptote at $y = -2$

$\Rightarrow D$

2. Geometry and Calculus, EXT1 2016 HSC 9 MC

When $x = 1$,

$f(1) < 1$

$f'(1) > 1$ (graph slope$^+$ $> 45°$)

$f''(1) < 1$ (concave down)

$\Rightarrow A$

3. Geometry and Calculus, EXT1 2012 HSC 13b

(i) $y = \dfrac{2x^2}{x^2+9}$

$= \dfrac{2}{1+\frac{9}{x^2}}$

As $x \to \infty,\ y \to 2$

As $x \to -\infty,\ y \to 2$

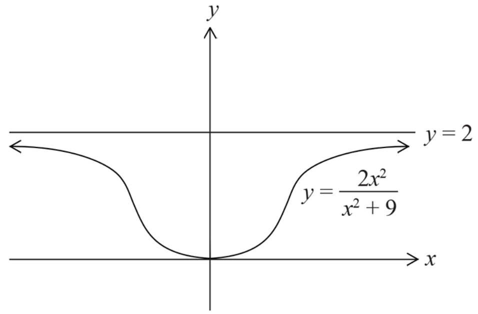
$\therefore$ Horizontal asymptote at $y = 2$

(ii) At $x = 0,\ y = 0$

$f(x) = \dfrac{2x^2}{x^2+9} \geq 0$ for all x

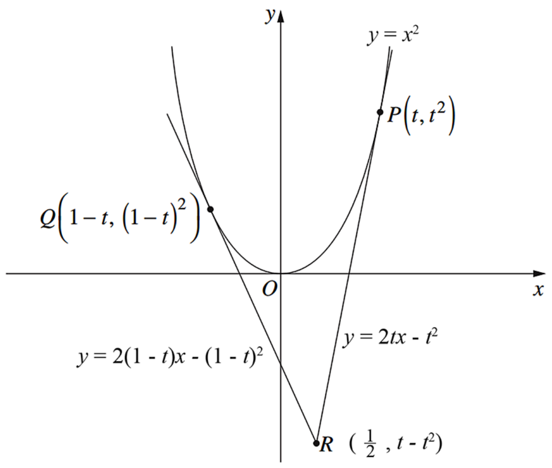
$f(-x) = \dfrac{2(-x)^2}{(-x)^2+9} = \dfrac{2x^2}{x^2+9} = f(x)$

Since $f(x) = f(-x) \Rightarrow$ EVEN function



4. Quadratic, EXT1 2011 HSC 3b

(i)



Show tangent at P is $y = 2tx - t^2$

$y = x^2$

$\dfrac{dy}{dx} = 2x$

$x = t$ at P

$\dfrac{dy}{dx} = 2t$

Find equation with $m = 2t$ through $P(t, t^2)$

$y - y_1 = m(x - x_1)$

$y - t^2 = 2t(x - t)$

$y = 2tx - 2t^2 + t^2$

$= 2tx - t^2$ ... as required

(ii) Tangent at Q has equation

$y = 2(1-t)x - (1-t)^2$

MARKER'S COMMENT: Many students derived this equation rather than substituting the new parameter, costing them valuable time. This is a benefit of

(iii) Need to show $R\left(\dfrac{1}{2},\ t - t^2\right)$

R is at intersection of tangents

$$2tx - t^2 = 2(1-t)x - (1-t)^2$$
$$2tx - t^2 = 2x - 2tx - 1 + 2t - t^2$$
$$4tx - 2x = -1 + 2t - t^2 + t^2$$
$$2x(2t-1) = 2t - 1$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Using $y = 2tx - t^2$ when $x = \frac{1}{2}$

$$y = 2t\left(\frac{1}{2}\right) - t^2$$
$$= t - t^2$$

$\therefore\ R\left(\frac{1}{2}, t - t^2\right)$ ... as required

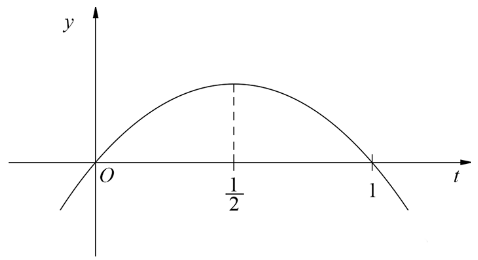
using the parametric approach.

(iv) Locus of R

Since $x = \frac{1}{2}$ is a constant

R is a vertical line

Now, $y = t - t^2 = t(1-t)$



Graphically, y has a maximum at $t = \frac{1}{2}$

Max $y = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow y < \frac{1}{4}$ (tangents can't meet on parabola)

$\therefore$ Locus of R is vertical line $x = \frac{1}{2},\ \ y < \frac{1}{4}$

♦♦ *Mean mark of 22%.*
MARKER'S COMMENT: Many students stated the locus as $y = t - t^2$ rather than realising it had to be a straight line since $x = ½$, and that $y = t - t^2$ simply restricted the values of y.

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5. Quadratic, EXT1 2006 HSC 2c

(i) U is the y intercept of PR

$$y = \frac{1}{2}(p+r)x - apr$$

When $x = 0$

$$y = -apr$$

$\therefore\ U$ has coordinates $(0, -apr)$

(ii) Show T is $(a(p+q), apq)$

Tangents at P and Q are

$$y = px - ap^2 \quad \ldots (1)$$
$$y = qx - aq^2 \quad \ldots (2)$$

T occurs when $(1) = (2)$

$$px - ap^2 = qx - aq^2$$
$$px - qx = ap^2 - aq^2$$
$$x(p-q) = a(p^2 - q^2)$$
$$= a(p-q)(p+q)$$
$$\therefore\ x = a(p+q)$$

Substitute $x = a(p+q)$ into (1)

$$y = p \cdot a(p+q) - ap^2$$
$$= ap^2 + apq - ap^2$$
$$= apq$$

$\therefore\ T(a(p+q), apq)$ … as required.

(iii) Axis of parabola $x^2 = 4ay$ is y-axis

$\Rightarrow TU$ will be perpendicular to the y-axis if it has a gradient of 0 (i.e. T and U have same y-values)

$\therefore$ Need to show $-apr = apq$.

Consider QR

Since it is perpendicular to y-axis and the parabola is symmetrical

$$\Rightarrow \quad 2aq = -2ar$$
$$q = -r$$
$$\therefore apq = ap(-r)$$
$$apq = -apr$$

$\therefore TU$ is perpendicular.

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6. Quadratic, EXT1 2007 HSC 5d

(i) Solution 1

Prove $p^2 + pq + 2 = 0$

$PQ \quad x + py - 2ap - ap^3 = 0$

$$\Rightarrow y = -\frac{1}{p}x + 2a + ap^2$$

$$\therefore m_{PQ} = -\frac{1}{p}$$

Also, $m_{PQ} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{ap^2 - aq^2}{2ap - 2aq}$$
$$= \frac{a(p+q)(p-q)}{2a(p-q)}$$
$$= \frac{p+q}{2}$$

$$\therefore \frac{p+q}{2} = -\frac{1}{p}$$
$$p^2 + pq = -2$$
$$p^2 + pq + 2 = 0 \quad \text{... as required}$$

Solution 2

$PQ \quad x + py - 2ap - ap^3 = 0$

$Q(2aq, aq^2)$ lies on PQ

$$\therefore 2aq + p(aq^2) - 2ap - ap^3 = 0$$
$$2q - 2p + pq^2 - p^3 = 0$$
$$2(q-p) + p(q^2 - p^2) = 0$$
$$2(q-p) + p(q-p)(q+p) = 0$$
$$p(q+p) + 2 = 0$$
$$p^2 + pq + 2 = 0 \quad \text{... as required}$$

(ii) If $OP \perp OQ$, show $p^2 = 2$

$$m_{OP} \times m_{OQ} = -1$$
$$m_{OP} = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2}$$
$$m_{OQ} = \frac{aq^2 - 0}{2aq - 0} = \frac{q}{2}$$
$$\therefore \frac{p}{2} \cdot \frac{q}{2} = -1$$
$$pq = -4$$

Substitute $pq = -4$ into part (i)

$$p^2 - 4 + 2 = 0$$
$$\therefore p^2 = 2 \quad \text{… as required}$$

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7. Quadratic, EXT1 2017 HSC 14b

(i) $x^2 = 4y, \quad \Rightarrow y = \dfrac{x^2}{4}$

$\dfrac{dy}{dx} = \dfrac{x}{2}$

At $P(2p, p^2)$,

$\dfrac{dy}{dx} = p$

Equation of tangent:

$y = px - p^2$

R and Q at intersection

$y = px - p^2 \ldots (1)$

$y = -\dfrac{x^2}{4a} \ldots (2)$

Subtract (1) − (2)

$px - p^2 + \dfrac{x^2}{4a} = 0$

$x^2 + 4apx - 4ap^2 = 0 \ldots$ as required

(ii) $x^2 + 4apx - 4ap^2 = 0$

♦♦ Mean mark 30%.

$$x = \frac{-4ap \pm \sqrt{(4ap)^2 - 4 \cdot 1 \cdot (-4ap^2)}}{2}$$

$$= \frac{-4ap \pm \sqrt{16ap^2(a+1)}}{2}$$

$$= -2ap \pm 2p\sqrt{a(a+1)}$$

$\Rightarrow$ x-coordinate of M is $-2ap$.

M lies on tangent $y = px - p^2$

$\therefore y = p(-2ap) - p^2$

$= -2ap^2 - p^2$

$= -p^2(2a+1)$

$\therefore M$ has coordinates $\left(-2ap, \ -p^2(2a+1)\right)$

(iii) If M always lies on $x^2 = -4y$

♦♦ Mean mark 23%.

$(-2ap)^2 = -4\left(-p^2(2a+1)\right)$

$4a^2p^2 = 4p^2(2a+1)$

$a^2 = 2a + 1$

$a^2 - 2a - 1 = 0$

$\therefore a = \dfrac{+2 \pm \sqrt{4 + 4 \cdot 1 \cdot 1}}{2}$

$= 1 \pm \sqrt{2}$

$= 1 + \sqrt{2}, \ a > 0$

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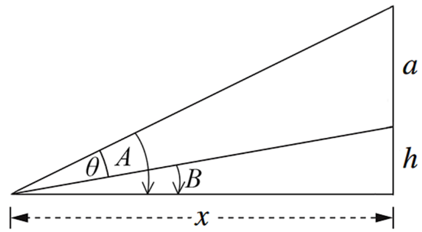
8. Geometry and Calculus, EXT1 2009 HSC 7b



♦♦ Mean mark data not available for parts of questions although Q7 as a whole scored <30%.

MARKER'S COMMENT: Answers that included a diagram and clearly labelled angles were generally successful.

Show $\theta = \tan^{-1}\left[\dfrac{ax}{x^2 + h(a+h)}\right]$

$\tan A = \dfrac{a+h}{x}$

$\tan B = \dfrac{h}{x}$

$\tan(A - B) = \dfrac{\frac{a+h}{x} - \frac{h}{x}}{1 + \left(\frac{a+h}{x}\right)\left(\frac{h}{x}\right)} \times \dfrac{x^2}{x^2}$

$$= \frac{x(a+h) - xh}{x^2 + h(a+h)}$$

$$= \frac{ax}{x^2 + h(a+h)}$$

Since $\theta = A - B$

$$\theta = \tan^{-1}\left[\frac{ax}{x^2 + h(a+h)}\right] \quad \text{... as required.}$$

(ii) Max when $\dfrac{d\theta}{dx} = 0$ and $x > 0$

Let $u = \dfrac{ax}{x^2 + h(a+h)}$

$$\theta = \tan^{-1} u$$

$$\frac{du}{dx} = \frac{a\left[x^2 + h(a+h)\right] - ax \cdot 2x}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{-ax^2 + ah(a+h)}{\left[x^2 + h(a+h)\right]^2}$$

$$\frac{d\theta}{dx} = \frac{d\theta}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$= \frac{1}{1 + \left[\frac{ax}{x^2+h(a+h)}\right]^2} \times \frac{-ax^2 + ah(a+h)}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{-ax^2 + ah(a+h)}{\left[x^2 + h(a+h)\right]^2 + a^2x^2}$$

Note that $\dfrac{d\theta}{dx} = 0$ when

$$-ax^2 + ah(a+h) = 0$$

$$ax^2 = ah(a+h)$$

$$x^2 = h(a+h)$$

$$x = \sqrt{h(a+h)} \quad (x > 0)$$