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The 2019 HSC exam is the last paper of the old course syllabus which provides a long history of high quality Mathematics exams.

In order to continue to add value to our subscribers, we have put together our “2019 HSC Comprehensive Revision Series” that we recommend motivated students aiming for a Band 5 or 6 result should **attempt, carefully review and annotate** in Term 3, creating a concise and high quality revision resource.

Note that our “Final HSC Revision Set” for students starting their revision later in the term will be available in early-September in the final stretch before the Extension 1 HSC exam on 4 November, 2019.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create an extremely important revision set that ensures students cover a wide cross-section of the key areas we have carefully identified.

IMPORTANT: If students have been exposed to many of the questions in these worksheets during the year, we say great! In sports vernacular, this is where cobwebs are turned into cables through repetition, confidence is built and speed through the paper is developed (an aspect we regard as critical to peak achievement).

[HSC Final Study: EXT1 Topics 9-10](#) (~12% historical contribution)

[Key Areas addressed by this worksheet](#)

Topic 9: The Parabola

- Super consistent contributor – numerous examples of multi-part “standard” questions containing a range of difficulty levels are reviewed. Chunky mark allocations and historically low mean marks make this topic revision crucial in achieving high band results!
- Locus has proven very challenging and is an important revision focus in 2019 (not tested for the last 3 years);
- Generation of both tangent and normal equation proofs (students should be able to do this – along with chord proof – *very quickly*. Not asked since 2011 but these proofs were common in the past).

Topic 10: Geometrical Applications of Calculus

- Curve sketching is a revision focus area as it has been under-examined in the last 5 years;
- The most common question type involving curves with asymptotes (vertical and horizontal) are reviewed;
- Max/min problems may also appear as a 2-3 mark part of a larger cross-topic question, as they did between 2012-2014 (this *does not include* any max/min elements of projectile or other Topic 14 questions). Note that this type of max/min problem (i.e. *ex-Topic 14*) has been absent in the last 4 exams and this revision set looks at relevant examples.

SmarterMaths HSC Teacher Edition

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~ Sean Donohue, Head Teacher, Nepean CAPA

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EXTENSION 1: 2019 HSC Revision Series

- T9 Quadratics and The Parabola EXT1
- T10 Geometrical Applications of Calculus
EXT1



Teacher: Smarter Maths

Exam Equivalent Time: 75 minutes (based on HSC allocation of 1.5 minutes approx. per mark)

Questions

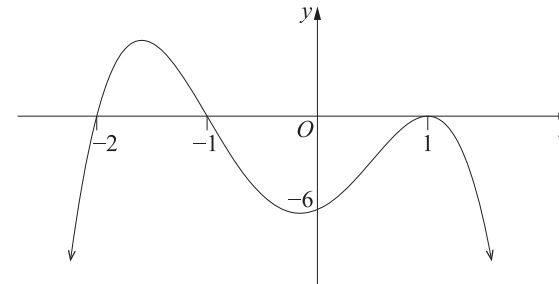
1. Geometry and Calculus, EXT1 2015 HSC 5 MC

What are the asymptotes of $y = \frac{3x}{(x+1)(x+2)}$

- (A) $y = 0$, $x = -1$, $x = -2$
- (B) $y = 0$, $x = 1$, $x = 2$
- (C) $y = 3$, $x = -1$, $x = -2$
- (D) $y = 3$, $x = 1$, $x = 2$

2. Geometry and Calculus, EXT1 2018 HSC 4 MC

The diagram shows the graph of $y = a(x+b)(x+c)(x+d)^2$.

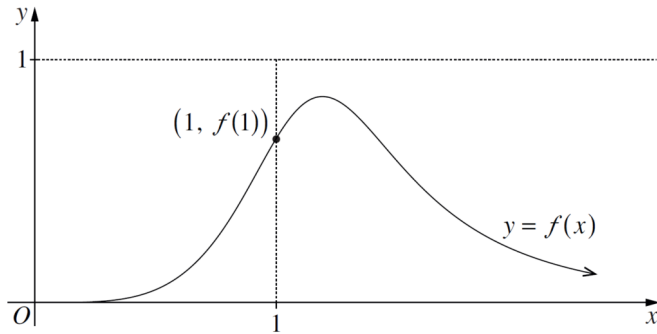


What are possible values of a , b , c and d ?

- A. $a = -6$, $b = -2$, $c = -1$, $d = 1$
- B. $a = -6$, $b = 2$, $c = 1$, $d = -1$
- C. $a = -3$, $b = -2$, $c = -1$, $d = 1$
- D. $a = -3$, $b = 2$, $c = 1$, $d = -1$

3. Geometry and Calculus, EXT1 2016 HSC 9 MC

The diagram shows the graph of $y = f(x)$.



Which of the following is a correct statement?

- (A) $f''(1) < f(1) < 1 < f'(1)$
- (B) $f''(1) < f'(1) < f(1) < 1$
- (C) $f(1) < 1 < f'(1) < f''(1)$
- (D) $f'(1) < f(1) < 1 < f''(1)$

4. Geometry and Calculus, EXT1 2012 HSC 13b

(i) Find the horizontal asymptote of the graph

$$y = \frac{2x^2}{x^2 + 9}. \quad (1 \text{ mark})$$

(ii) Without the use of calculus, sketch the graph

$$y = \frac{2x^2}{x^2 + 9}, \text{ showing the asymptote found in part (i)}. \quad (2 \text{ marks})$$

5. Quadratic, EXT1 2015 HSC 12b

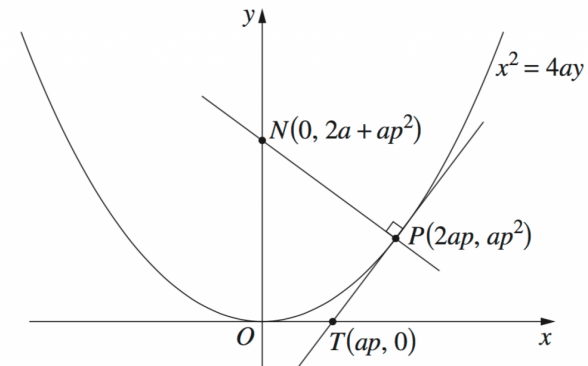
The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The equation of the chord PQ is given by $(p + q)x - 2y - 2apq = 0$. (Do NOT prove this.)

- (i) Show that if PQ is a focal chord then $pq = -1$. (1 mark)
- (ii) If PQ is a focal chord and P has coordinates $(8a, 16a)$, what are the coordinates of Q in terms of a ? (2 marks)

6. Quadratic, EXT1 2013 HSC 13b

The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The tangent to the parabola at P meets the x -axis at $T(ap, 0)$. The normal to the tangent at P meets the y -axis at $N(0, 2a + ap^2)$.

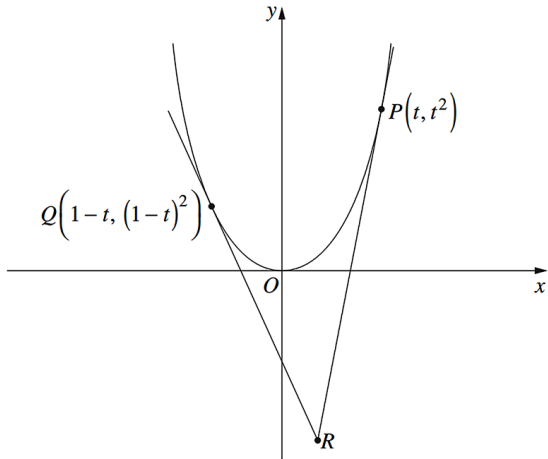


The point G divides NT externally in the ratio $2 : 1$.

- (i) Show that the coordinates of G are $(2ap, -2a - ap^2)$. (2 marks)
- (ii) Show that G lies on a parabola with the same directrix and focal length as the original parabola. (2 marks)

7. Quadratic, EXT1 2011 HSC 3b

The diagram shows two distinct points $P(t, t^2)$ and $Q(1-t, (1-t)^2)$ on the parabola $y = x^2$. The point R is the intersection of the tangents to the parabola at P and Q .



- (i) Show that the equation of the tangent to the parabola at P is $y = 2tx - t^2$. (2 marks)
- (ii) Using part (i), write down the equation of the tangent to the parabola at Q . (1 mark)
- (iii) Show that the tangents at P and Q intersect at $R\left(\frac{1}{2}, t - t^2\right)$. (2 marks)
- (iv) Describe the locus of R as t varies, stating any restriction on the y -coordinate. (2 marks)

8. Geometry and Calculus, EXT1 2009 HSC 4b

Consider the function $f(x) = \frac{x^4 + 3x^2}{x^4 + 3}$.

- (i) Show that $f(x)$ is an even function. (1 mark)
- (ii) What is the equation of the horizontal asymptote to the graph $y = f(x)$? (1 mark)
- (iii) Find the x -coordinates of all stationary points for the graph $y = f(x)$. (3 marks)
- (iv) Sketch the graph $y = f(x)$. You are not required to find any points of inflexion. (2 marks)

9. Quadratic, EXT1 2005 HSC 4c

The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

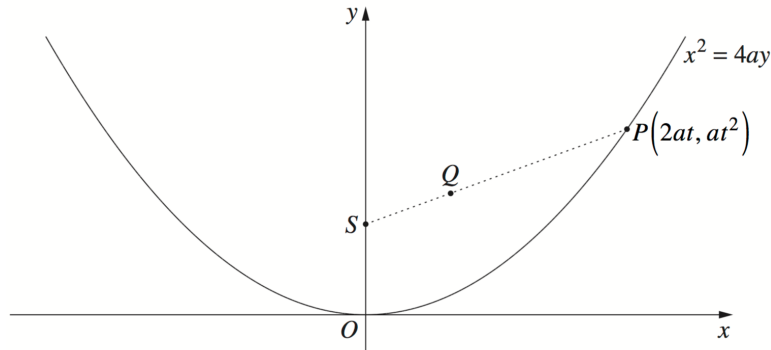
The equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and the equation of the normal at Q is similarly given by $x + qy = 2aq + aq^3$.

- (i) Show that the normals at P and Q intersect at the point R whose coordinates are $(-apq[p + q], a[p^2 + pq + q^2 + 2])$. (2 marks)
- (ii) The equation of the chord PQ is $y = \frac{1}{2}(p + q)x - apq$. (Do NOT show this.)
If the chord PQ passes through $(0, a)$, show that $pq = -1$. (1 mark)
- (iii) Find the equation of the locus of R if the chord PQ passes through $(0, a)$. (2 marks)

10. Quadratic, EXT1 2014 HSC 13c

The point $P(2at, at^2)$ lies on the parabola $x^2 = 4ay$ with focus S .

The point Q divides the interval PS internally in the ratio $t^2 : 1$.



(i) Show that the coordinates of Q are

$$x = \frac{2at}{1+t^2} \text{ and } y = \frac{2at^2}{1+t^2}. \text{ (2 marks)}$$

(ii) Express the slope of OQ in terms of t . (1 mark)

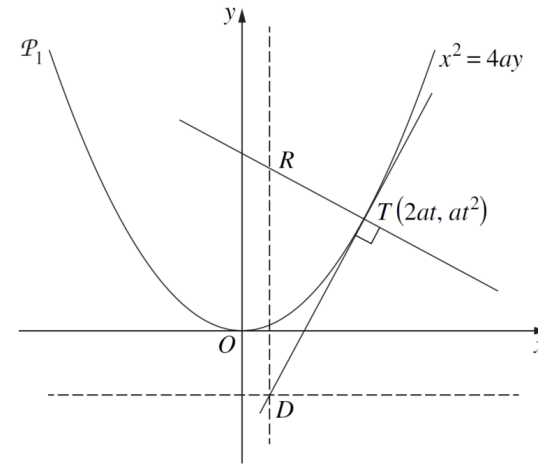
(iii) Using the result from part (ii), or otherwise, show that Q lies on a fixed circle of radius a . (3 marks)

11. Quadratic, EXT1 2016 HSC 14c

The point $T(2at, at^2)$ lies on the parabola P_1 with the equation $x^2 = 4ay$.

The tangent to the parabola P_1 at T meets the directrix at D .

The normal to the parabola P_1 at T meets the vertical line through D at the point R , as shown in the diagram.



i. Show that the point D has coordinates $(at - \frac{a}{t}, -a)$. (1 mark)

ii. Show that the locus of R lies on another parabola P_2 . (3 marks)

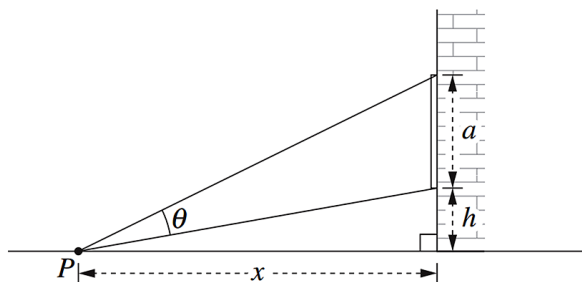
iii. State the focal length of the parabola P_2 . (1 mark)

It can be shown that the minimum distance between R and T occurs when the normal to P_1 at T is also the normal to P_2 at R . (Do NOT prove this.)

iv. Find the values of t so that the distance between R and T is a minimum. (2 marks)

12. Geometry and Calculus, EXT1 2009 HSC 7b

A billboard of height a metres is mounted on the side of a building, with its bottom edge h metres above street level. The billboard subtends an angle θ at the point P , x metres from the building.



- (i) Use the identity $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ to show that

$$\theta = \tan^{-1} \left[\frac{ax}{x^2 + h(a + h)} \right]. \quad (2 \text{ marks})$$

- (ii) The maximum value of θ occurs when $\frac{d\theta}{dx} = 0$ and x is positive.

Find the value of x for which θ is a maximum. (3 marks)

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Worked Solutions

1. Geometry and Calculus, EXT1 2015 HSC 5 MC

$$y = \frac{3x}{(x + 1)(x + 2)}$$

Asymptotes at $x = -1$ and $x = -2$

As $x \rightarrow \infty$, $y \rightarrow 0^+$

As $x \rightarrow -\infty$, $y \rightarrow 0^-$

\therefore Horizontal asymptote at $y = 0$

$\Rightarrow A$

2. Geometry and Calculus, EXT1 2018 HSC 4 MC

Single roots at: $x = -2, -1$

Double root at: $x = 1$

$\therefore b = 2, c = 1, d = -1$

y -intercept occurs when $x = 0$,

$\Rightarrow abcd^2 = -6$

$a(2)(1)(1^2) = -6$

$\therefore a = -3$

$\Rightarrow D$

3. Geometry and Calculus, EXT1 2016 HSC 9 MC

When $x = 1$,

$$f(1) < 1$$

$$f'(1) > 1 \text{ (graph slope}^+ > 45^\circ)$$

$$f''(1) < 1 \text{ (concave down)}$$

$\Rightarrow A$

4. Geometry and Calculus, EXT1 2012 HSC 13b

$$\begin{aligned} \text{(i) } y &= \frac{2x^2}{x^2 + 9} \\ &= \frac{2}{1 + \frac{9}{x^2}} \end{aligned}$$

As $x \rightarrow \infty$, $y \rightarrow 2$

As $x \rightarrow -\infty$, $y \rightarrow 2$

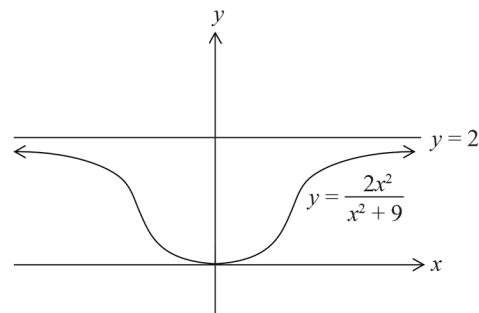
\therefore Horizontal asymptote at $y = 2$

(ii) At $x = 0$, $y = 0$

$$f(x) = \frac{2x^2}{x^2 + 9} \geq 0 \text{ for all } x$$

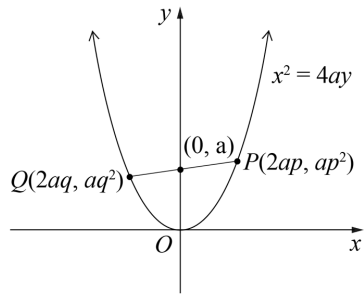
$$f(-x) = \frac{2(-x)^2}{(-x)^2 + 9} = \frac{2x^2}{x^2 + 9} = f(x)$$

Since $f(x) = f(-x) \Rightarrow$ EVEN function



5. Quadratic, EXT1 2015 HSC 12b

(i) Show if PQ is a focal chord $pq = -1$



$$(p + q)x - 2y - 2apq = 0$$

If PQ is a focal chord, it passes $(0, a)$

$$(p + q)0 - 2a - 2apq = 0$$

$$2apq = -2a$$

$$pq = \frac{-2a}{2a}$$

$$= -1 \dots \text{as required}$$

(ii) If $P(2ap, ap^2) = (8a, 16a)$

$$\Rightarrow 2ap = 8a$$

$$p = 4$$

Using $q = -\frac{1}{p}$ (from part (i))

$$q = -\frac{1}{4}$$

$$\begin{aligned} \therefore Q(2aq, aq^2) &= \left(2a\left(-\frac{1}{4}\right), a\left(-\frac{1}{4}\right)^2\right) \\ &= \left(-\frac{a}{2}, \frac{a}{16}\right) \end{aligned}$$

6. Quadratic, EXT1 2013 HSC 13b

(i) $N(0, 2a + ap^2)$

$T(ap, 0)$

G divides NT externally in ratio $2:1$,

i.e. $(m:n = 2:-1)$

$$\begin{aligned} \therefore G &= \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right) \\ &= \left(\frac{0 + 2ap}{2-1}, \frac{-2a - ap^2 + 0}{2-1}\right) \\ &= (2ap, -2a - ap^2) \dots \text{as required} \end{aligned}$$

(ii) $x^2 = 4ay$ has focal length a

and directrix $y = -a$

Locus of G

$$x = 2ap \dots (1)$$

$$y = -2a - ap^2 \dots (2)$$

$$\text{From (1), } p = \frac{x}{2a}$$

Substitute $p = \frac{x}{2a}$ into (2)

$$y = -2a - a\left(\frac{x}{2a}\right)^2$$

$$y = -2a - \frac{ax^2}{4a^2}$$

$$\frac{x^2}{4a} = -y - 2a$$

$$x^2 = -4a(y + 2a)$$

Focal length $= a$

Vertex at $(0, -2a)$

Directrix $y = -2a + a = -a$

\therefore Locus of G has same focal length and

directrix as $x^2 = 4ay$.

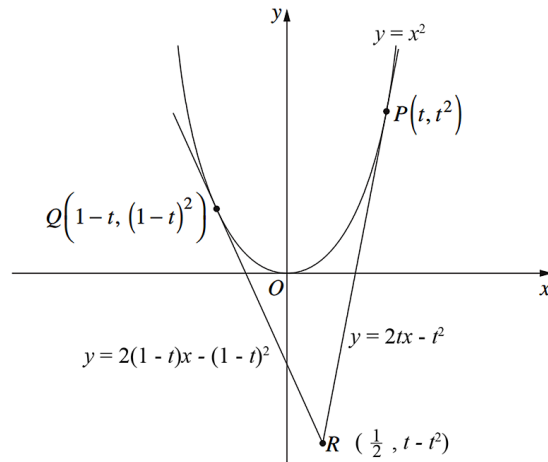
♦♦ Mean mark 31%.

IMPORTANT: Remember that finding the locus involves eliminating the parameter.

Expressing the locus in the form $x^2 = 4ay$ is critical to finding its directrix and focal length.

7. Quadratic, EXT1 2011 HSC 3b

(i)



Show tangent at P is $y = 2tx - t^2$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$x = t \text{ at } P$$

$$\frac{dy}{dx} = 2t$$

Find equation with $m = 2t$ through $P(t, t^2)$

$$y - y_1 = m(x - x_1)$$

$$y - t^2 = 2t(x - t)$$

$$y = 2tx - 2t^2 + t^2$$

$$= 2tx - t^2 \dots \text{ as required}$$

(ii) Tangent at Q has equation

$$y = 2(1-t)x - (1-t)^2$$

(iii) Need to show $R\left(\frac{1}{2}, t - t^2\right)$

MARKER'S COMMENT: Many students derived this equation rather than substituting the new parameter, costing them valuable time. This is a benefit of

R is at intersection of tangents

$$2tx - t^2 = 2(1-t)x - (1-t)^2$$

$$2tx - t^2 = 2x - 2tx - 1 + 2t - t^2$$

$$4tx - 2x = -1 + 2t - t^2 + t^2$$

$$2x(2t - 1) = 2t - 1$$

$$2x = 1$$

$$x = \frac{1}{2}$$

Using $y = 2tx - t^2$ when $x = \frac{1}{2}$

$$y = 2t\left(\frac{1}{2}\right) - t^2$$

$$= t - t^2$$

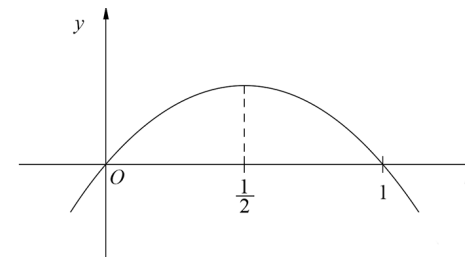
$$\therefore R\left(\frac{1}{2}, t - t^2\right) \dots \text{ as required}$$

(iv) Locus of R

Since $x = \frac{1}{2}$ is a constant

R is a vertical line

Now, $y = t - t^2 = t(1 - t)$



Graphically, y has a maximum at $t = \frac{1}{2}$

$$\text{Max } y = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\Rightarrow y < \frac{1}{4} \text{ (tangents can't meet on parabola)}$$

using the parametric approach.

♦♦ Mean mark of 22%.

MARKER'S COMMENT: Many students stated the locus as $y = t - t^2$ rather than realising it had to be a straight line since $x = \frac{1}{2}$, and that $y = t - t^2$ simply restricted the values of y .

\therefore Locus of R is vertical line $x = \frac{1}{2}$, $y < \frac{1}{4}$

8. Geometry and Calculus, EXT1 2009 HSC 4b

(i) $f(x) = \frac{x^4 + 3x^2}{x^4 + 3}$

$$\begin{aligned} f(-x) &= \frac{(-x)^4 + 3(-x)^2}{(-x)^4 + 3} \\ &= \frac{x^4 + 3x^2}{x^4 + 3} \\ &= f(x) \end{aligned}$$

\therefore Even function.

(ii) $y = \frac{x^4 + 3x^2}{x^4 + 3}$

$$= \frac{1 + \frac{3}{x^2}}{1 + \frac{3}{x^4}}$$

As $x \rightarrow \infty$
 $y \rightarrow 1$

\therefore Horizontal asymptote at $y = 1$

(iii) $f(x) = \frac{x^4 + 3x^2}{x^4 + 3}$

$u = x^4 + 3x^2 \quad v = x^4 + 3$

$u' = 4x^3 + 6x \quad v' = 4x^3$

$f'(x) = \frac{u'v - uv'}{v^2}$

$$= \frac{(4x^3 + 6x)(x^4 + 3) - (x^4 + 3x^2)4x^3}{(x^4 + 3)^2}$$

$$\begin{aligned} &= \frac{4x^7 + 12x^3 + 6x^5 + 18x - 4x^7 - 12x^5}{(x^4 + 3)^2} \\ &= \frac{-6x^5 + 12x^3 + 18x}{(x^4 + 3)^2} \\ &= \frac{-6x(x^4 - 2x^2 - 3)}{(x^4 + 3)^2} \end{aligned}$$

S.P. when $x = 0$ or $x^4 - 2x^2 - 3 = 0$

Let $X = x^2$

$X^2 - 2X - 3 = 0$

$(X - 3)(X + 1) = 0$

$X = 3$ or -1

$\therefore x^2 = 3$ or $x^2 = -1$

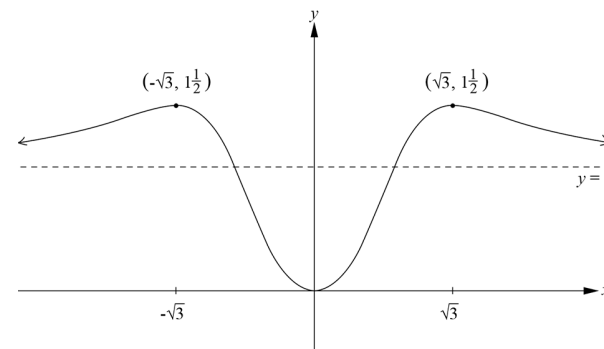
$x = \pm\sqrt{3}$ (no solution)

\therefore SPs occur when $x = 0, -\sqrt{3}, \sqrt{3}$

(iv) When $x = 0, y = 0$

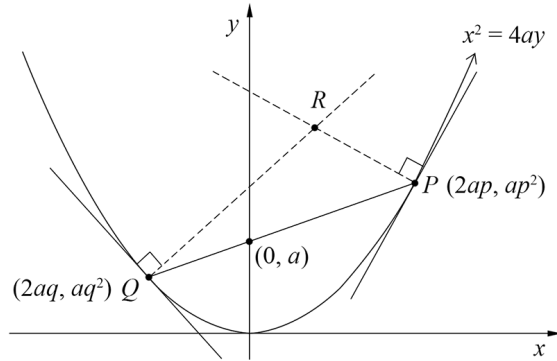
When $x = \sqrt{3}, y = \frac{(\sqrt{3})^4 + 3(\sqrt{3})^2}{(\sqrt{3})^4 + 3} = \frac{3}{2}$

MARKER'S COMMENT: Many students did not realise the denominator of $f'(x)$ could be ignored when equating $f'(x) = 0$.



9. Quadratic, EXT1 2005 HSC 4c

(i) Show R is $(-apq[p + q], a[p^2 + pq + q^2 + 2])$



Equations of normals through P and Q

$$x + py = 2ap + ap^3 \dots (1)$$

$$x + qy = 2aq + aq^3 \dots (2)$$

Multiply (1) $\times q$, (2) $\times p$

$$qx + pqy = 2apq + ap^3q \dots (3)$$

$$px + pqy = 2apq + apq^3 \dots (4)$$

Subtract (4) - (3)

$$\begin{aligned} x(p - q) &= apq^3 - ap^3q \\ &= apq(q^2 - p^2) \\ &= apq(q - p)(q + p) \\ &= -apq(p - q)(p + q) \\ x &= -apq[p + q] \end{aligned}$$

Substitute $x = -apq[p + q]$ into (1)

$$\begin{aligned} py - apq[p + q] &= 2ap + ap^3 \\ py &= 2ap + ap^3 + apq(p + q) \\ y &= 2a + ap^2 + aq(p + q) \\ &= 2a + ap^2 + apq + aq^2 \\ &= a[p^2 + pq + q^2 + 2] \end{aligned}$$

$\therefore R$ is $(-apq[p + q], a[p^2 + pq + q^2 + 2])$
... as required.

(ii) PQ is $y = \frac{1}{2}(p + q)x - apq$

Passes $(0, a)$

$$a = \frac{1}{2}(p + q)0 - apq$$

$$a = -apq$$

$\therefore pq = -1$... as required

(iii) R has coordinates

$$x = -apq[p + q]$$

$$x = a(p + q) \text{ (using part (ii))}$$

$$\frac{x}{a} = (p + q)$$

$$\begin{aligned} y &= a[p^2 + pq + q^2 + 2] \\ &= a(p^2 - 1 + q^2 + 2) \\ &= a(p^2 + q^2 + 1) \\ &= a[(p + q)^2 - 2pq + 1] \\ &= a[(p + q)^2 + 3] \\ &= a\left[\left(\frac{x}{a}\right)^2 + 3\right] \end{aligned}$$

$$= \frac{x^2}{a} + 3a$$

$$\therefore \text{Locus of } R \text{ is } y = \frac{x^2}{a} + 3a$$

10. Quadratic, EXT1 2014 HSC 13c

(i) Show $Q = \left(\frac{2at}{1+t^2}, \frac{2at^2}{1+t^2} \right)$

$$P(2at, at^2), S(0, a)$$

PS is divided internally in ratio $t^2 : 1$

$$\begin{aligned} Q &= \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n} \right) \\ &= \left(\frac{1(2at) + t^2(0)}{t^2 + 1}, \frac{1(at^2) + t^2(a)}{t^2 + 1} \right) \\ &= \left(\frac{2at}{1+t^2}, \frac{2at^2}{1+t^2} \right) \dots \text{as required.} \end{aligned}$$

(ii) $m_{OQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$$\begin{aligned} &= \frac{\frac{2at^2}{1+t^2}}{\frac{2at}{1+t^2}} \times \frac{1+t^2}{1+t^2} \\ &= \frac{2at^2}{2at} \\ &= t \end{aligned}$$

(iii) Show Q lies on a fixed circle radius a

Since Q passes through $(0, 0)$

\Rightarrow If locus of Q is a circle, it has diameter QT where $T(0, 2a)$

Show $QT \perp OQ$

(angles on circum. subtended by a diameter are 90°)

♦♦ Mean mark 22%

$$m_{OQ} = t \quad (\text{see part (ii)})$$

Find m_{QT} , where:

$$Q \left(\frac{2at}{1+t^2}, \frac{2at^2}{1+t^2} \right), \quad T(0, 2a)$$

$$\begin{aligned} m_{QT} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\frac{2at^2}{1+t^2} - 2a}{\frac{2at}{1+t^2} - 0} \\ &= \frac{2at^2 - 2a(1+t^2)}{2at} \\ &= -\frac{2a}{2at} \\ &= -\frac{1}{t} \end{aligned}$$

$$m_{QT} \times m_{OQ} = -\frac{1}{t} \times t = -1$$

$$\Rightarrow QT \perp OQ$$

$\Rightarrow O, T, Q$ lie on a circle.

\therefore Locus of Q is a fixed circle, centre $(0, a)$, radius a

11. Quadratic, EXT1 2016 HSC 14c

i. Show $D \left(at - \frac{a}{t}, -a \right)$

Tangent equation at T :

$$y = tx - at^2$$

D occurs when $y = -a$,

$$tx - at^2 = -a$$

$$tx = at^2 - a$$

$$x = at - \frac{a}{t}$$

$\therefore D$ has coordinates $\left(at - \frac{a}{t}, -a\right)$

ii. Normal equation at T :

$$x + ty = 2at + at^3$$

R occurs when $x = at - \frac{a}{t}$

$$at - \frac{a}{t} + ty = 2at + at^3$$

$$ty = at + \frac{a}{t} + at^3$$

$$y = a + \frac{a}{t^2} + at^2$$

$$= a\left(1 + \frac{1}{t^2} + t^2\right) \dots (*)$$

$$\therefore R\left(a\left(t - \frac{1}{t}\right), a\left(1 + \frac{1}{t^2} + t^2\right)\right)$$

$$x^2 = a^2\left(t - \frac{1}{t}\right)^2$$

$$= a^2\left(t^2 - 2 + \frac{1}{t^2}\right)$$

$$= a + \left(1 + \frac{1}{t^2} + t^2 - 3\right)$$

$$= a^2\left(\frac{y}{a} - 3\right) \text{ (see (*) above)}$$

$$= ay - 3a^2$$

\therefore Locus of R is $x^2 = ay - 3a^2$

iii. In the form $x^2 = 4ay$,

$$x^2 = a(y - 3a)$$

$$= 4 \cdot \frac{a}{4}(y - 3a)$$

\therefore Focal length = $\frac{a}{4}$

iv. Equation of P_2

$$x^2 = ay - 3a^2$$

$$y = \frac{x^2}{a} + 3a$$

$$y' = \frac{2x}{a}$$

At R , $x = a\left(t - \frac{1}{t}\right)$

\therefore Gradient of normal at R

$$\begin{aligned} &= \frac{-a}{2x} \\ &= \frac{-a}{2a\left(t - \frac{1}{t}\right)} \times \frac{t}{t} \\ &= \frac{-t}{2(t^2 - 1)} \end{aligned}$$

Gradient of normal at T :

$$x + ty = 2at + at^3$$

$$y = -\frac{1}{t}x + 2a + at^2$$

$$\therefore m = -\frac{1}{t}$$

Distance RT is a minimum when

$$\frac{-t}{2(t^2 - 1)} = -\frac{1}{t}$$

$$t^2 = 2t^2 - 2$$

◆◆◆ Mean mark 15%.

◆◆◆ Mean mark 11%.

◆◆ Mean mark 32%.

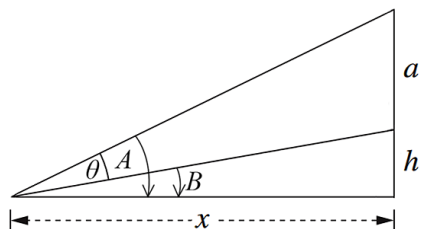
MARKER'S COMMENT:

Normal equation is found in the Reference Sheet. Save time by using it!

$$t^2 = 2$$

$$\therefore t = \pm \sqrt{2}$$

12. Geometry and Calculus, EXT1 2009 HSC 7b



♦♦ Mean mark data not available for parts of questions although Q7 as a whole scored <30%.

MARKER'S

COMMENT: Answers that included a diagram and clearly labelled angles were generally successful.

$$\text{Show } \theta = \tan^{-1} \left[\frac{ax}{x^2 + h(a+h)} \right]$$

$$\tan A = \frac{a+h}{x}$$

$$\tan B = \frac{h}{x}$$

$$\begin{aligned} \tan(A - B) &= \frac{\frac{a+h}{x} - \frac{h}{x}}{1 + \left(\frac{a+h}{x}\right)\left(\frac{h}{x}\right)} \times \frac{x^2}{x^2} \\ &= \frac{x(a+h) - xh}{x^2 + h(a+h)} \\ &= \frac{ax}{x^2 + h(a+h)} \end{aligned}$$

Since $\theta = A - B$

$$\theta = \tan^{-1} \left[\frac{ax}{x^2 + h(a+h)} \right] \dots \text{ as required.}$$

(ii) Max when $\frac{d\theta}{dx} = 0$ and $x > 0$

$$\text{Let } u = \frac{ax}{x^2 + h(a+h)}$$

$$\theta = \tan^{-1} u$$

$$\begin{aligned} \frac{du}{dx} &= \frac{a[x^2 + h(a+h)] - ax \cdot 2x}{[x^2 + h(a+h)]^2} \\ &= \frac{-ax^2 + ah(a+h)}{[x^2 + h(a+h)]^2} \end{aligned}$$

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{d\theta}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{1+u^2} \cdot \frac{du}{dx} \\ &= \frac{1}{1 + \left[\frac{ax}{x^2 + h(a+h)}\right]^2} \times \frac{-ax^2 + ah(a+h)}{[x^2 + h(a+h)]^2} \\ &= \frac{-ax^2 + ah(a+h)}{[x^2 + h(a+h)]^2 + a^2x^2} \end{aligned}$$

Note that $\frac{d\theta}{dx} = 0$ when

$$-ax^2 + ah(a+h) = 0$$

$$ax^2 = ah(a+h)$$

$$x^2 = h(a+h)$$

$$x = \sqrt{h(a+h)} \quad (x > 0)$$