√SmarterMaths

Thank you for subscribing to SmarterMaths Teacher Edition (Silver) in 2018.

The "HSC Final Stretch Revision Series" is the more concise version of our "HSC Comprehensive Revision Series" (released in Term 3) for students aiming for Band 5 and 6 results and are looking for around 6 hours of highly targeted revision in the last week before the HSC exam.

IMPORTANT: If students did complete the longer "HSC Comprehensive Revision Series" in Term 3, this can also be a revision set for you. Many of the higher performing students in past HSC's attest to doing past exams ***twice*** as a secret to their success. This type of revision repetition builds confidence and *speed through the paper*, both critical to peak achievement.

Our analysis on each topic, the common question types, past areas of difficulty and recent HSC trends all combine to create a super effective "final stretch" revision series.

HSC Final Study: EXT1 Topic 14 (~ 24% historical contribution)

Key Areas addressed by this worksheet

Topic 14: Other Motion (~ 2.5% historical contribution)

- covers motion outside of the major areas of projectile and simple harmonic motion;
- good mark allocations in 3 of the last 4 years;

Topic 14: Rates of Change (~ 4% historical contribution)

- *key revision focus* - asked in 9 of the last 10 years; 2017 exam's 1-mark allocation an outlier with previous questions all worth between 3-10 marks;
- underlying themes: area/volume questions, Pythagoras and other trig;
- constant rate of change (caused problems in the past).

Topic 14: Exponential Growth and Decay (~ 3.5% historical contribution)

- asked in 7 of the last 8 years; highest mean marks of any topic 14 sub-category;
- standard proof of rate of growth differential equation (regularly asked in early part of questions);

Extension 1 Mathematics
HSC "Final Stretch" Revision Series

14. Calculus and the Physical World EXT1
Exponential Growth and Decay EXT1
Other Motion EXT1
Rates of Change EXT1

Teacher: SmarterMaths
Exam Equivalent Time: 45 minutes *(based on HSC allocation of 1.5 minutes approx. per mark)*

Questions

1. Calculus in the Physical World, EXT1 2015 HSC 2 MC

Given that $N = 100 + 80e^{kt}$, which expression is equal to $\dfrac{dN}{dt}$?

(A) $k(100 - N)$

(B) $k(180 - N)$

(C) $k(N - 100)$

(D) $k(N - 180)$

---

2. Calculus in the Physical World, EXT1 2016 HSC 7 MC

The displacement x of a particle at time t is given by

$$x = 5\sin 4t + 12\cos 4t.$$

What is the maximum velocity of the particle?

(A) 13

(B) 28

(C) 52

(D) 68

---

3. Calculus in the Physical World, EXT1 2010 HSC 2b

The mass M of a whale is modelled by

$$M = 36 - 35.5e^{-kt}$$

where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

(i) Show that the rate of growth of the mass of the whale is given by the differential equation
$\dfrac{dM}{dt} = k(36 - M)$ *(1 mark)*

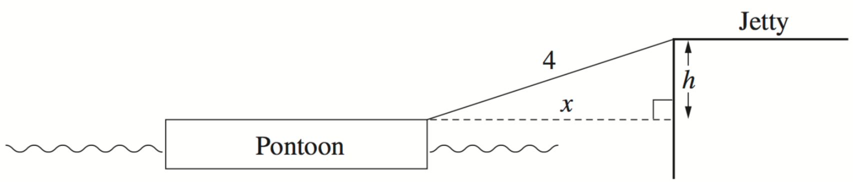
(ii) When the whale is 10 years old its mass is 20 tonnes.
Find the value of k, correct to three decimal places. *(2 marks)*

(iii) According to this model, what is the limiting mass of the whale? *(1 mark)*

---

4. Calculus in the Physical World, EXT1 2004 HSC 3c

A ferry wharf consists of a floating pontoon linked to a jetty by a 4 metre long walkway. Let h metres be the difference in height between the top of the pontoon and the top of the jetty and let x metres be the horizontal distance between the pontoon and the jetty.



(i) Find an expression for x in terms of h. *(1 mark)*

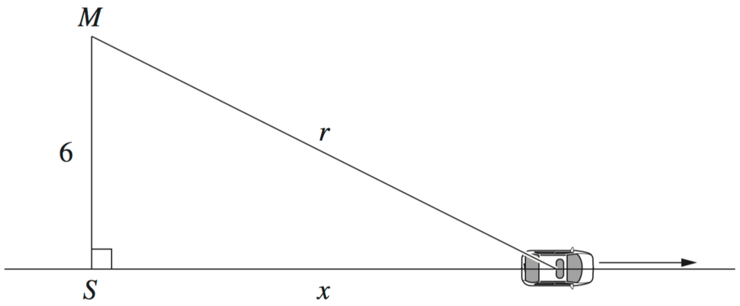
(ii) When the top of the pontoon is 1 metre lower than the top of the jetty, the tide is rising at a rate of 0.3 metres per hour.
At what rate is the pontoon moving away from the jetty? *(3 marks)*

---

5. Calculus in the Physical World, EXT1 2010 HSC 2d

A radio transmitter M is situated $6\ \text{km}$ from a straight road. The closest point on the road to the transmitter is S.

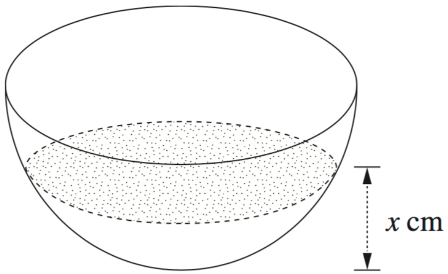
A car is travelling away from S along the road at a speed of $100\ \text{km h}^{-1}$. The distance from the car to S is $x\ \text{km}$ and from the car to M is $r\ \text{km}$.



Find an expression in terms of x for $\dfrac{dr}{dt}$, where t is time in hours. *(3 marks)*

---

6. Calculus in the Physical World, EXT1 2006 HSC 5c



A hemispherical bowl of radius $r\ \text{cm}$ is initially empty. Water is poured into it at a constant rate of $k\ \text{cm}^3$ per minute. When the depth of water in the bowl is $x\ \text{cm}$, the volume, $V\ \text{cm}^3$, of water in the bowl is given by

$$V = \frac{\pi}{3}x^2(3r - x). \qquad \text{(Do NOT prove this)}$$

(i) Show that

$$\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}. \quad \textit{(2 marks)}$$

(ii) Hence, or otherwise, show that it takes 3.5 times as long to fill the bowl to the point where $x = \frac{2}{3}r$ as it does to fill the bowl to the point where $x = \frac{1}{3}r$. *(2 marks)*

---

7. Calculus in the Physical World, EXT1 2013 HSC 13a

A spherical raindrop of radius r metres loses water through evaporation at a rate that depends on its surface area. The rate of change of the volume V of the raindrop is given by

$$\frac{dV}{dt} = -10^{-4}A,$$

where t is time in seconds and A is the surface area of the raindrop. The surface area and the volume of the raindrop are given by $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ respectively.

(i) Show that $\frac{dr}{dt}$ is constant. *(1 mark)*

(ii) How long does it take for a raindrop of volume 10^{-6} m^3 to completely evaporate? *(2 marks)*

---

8. Calculus in the Physical World, EXT1 2004 HSC 5a

A particle is moving along the x-axis, starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of $5\ \text{ms}^{-1}$ and an acceleration given by

$$\ddot{x} = 2x^3 + 2x.$$

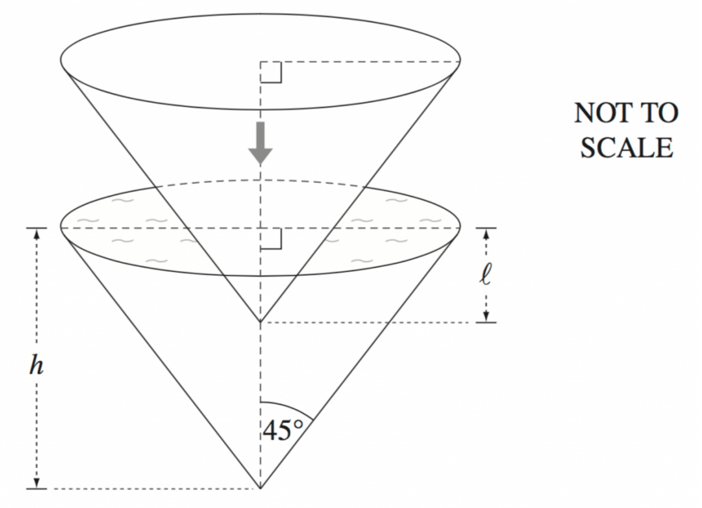
(i) Show that $\dot{x} = x^2 + 1$. *(2 marks)*

(ii) Hence find an expression for x in terms of t. *(3 marks)*

---

9. Calculus in the Physical World, EXT1 2011 HSC 7a

The diagram shows two identical circular cones with a common vertical axis. Each cone has height h cm and semi-vertical angle $45°$.



The lower cone is completely filled with water. The upper cone is lowered vertically into the water as shown in the diagram. The rate at which it is lowered is given by

$$\frac{dl}{dt} = 10,$$

where l cm is the distance the upper cone has descended into the water after t seconds.

As the upper cone is lowered, water spills from the lower cone. The volume of water remaining in the lower cone at time t is V cm³.

(i) Show that

$$V = \frac{\pi}{3}\left(h^3 - l^3\right).$$ *(1 mark)*

(ii) Find the rate at which V is changing with respect to time when $l = 2$. *(2 marks)*

(iii) Find the rate at which V is changing with respect to time when the lower cone has lost $\frac{1}{8}$ of its water. Give your answer in terms of h. *(2 marks)*

Worked Solutions

1. Calculus in the Physical World, EXT1 2015 HSC 2 MC

$$N = 100 + 80e^{kt}$$

$$\frac{dN}{dt} = k \times 80e^{kt}$$

$$= k(N - 100)$$

$$\Rightarrow C$$

---

2. Calculus in the Physical World, EXT1 2016 HSC 7 MC

$$x = 5\sin 4t + 12\cos 4t$$

$$\frac{dx}{dt} = 20\cos 4t - 48\sin 4t$$

$\Rightarrow$ Can be written in the form:

$A\cos(4t + \alpha)$, where

$$A = \sqrt{20^2 + 48^2}$$

$$= 52$$

$\therefore$ Max $v = 52\text{ ms}^{-1}$

$\Rightarrow \ C$

Worked Solutions

3. Calculus in the Physical World, EXT1 2010 HSC 2b

(i) $M = 36 - 35.5e^{-kt}$

$$35.5e^{-kt} = 36 - M$$

$$\therefore \frac{dM}{dt} = -k \times -35.5e^{-kt}$$

$$= k \times 35.5e^{-kt}$$

$$= k(36 - M) \quad \text{... as required}$$

IMPORTANT: Students must be well practised in this standard proof and be able to produce it quickly.

(ii) Find k

When $t = 10$, $M = 20$

$$M = 36 - 35.5e^{-kt}$$

$$20 = 36 - 35.5e^{-10k}$$

$$35.5e^{-10k} = 16$$

$$\ln e^{-10k} = \ln\left(\frac{16}{35.5}\right)$$

$$-10k = \ln\left(\frac{16}{35.5}\right)$$

$$\therefore k = -\frac{\ln\left(\frac{16}{35.5}\right)}{10}$$

$$= 0.07969...$$

$$= 0.080 \text{ (to 3 d.p.)}$$

(iii) As $t \to \infty$, $e^{-kt} = \dfrac{1}{e^{kt}} \to 0, \quad k > 0$

$$M \to 36$$

$\therefore$ The whale's limiting mass is 36 tonnes.

---

4. Calculus in the Physical World, EXT1 2004 HSC 3c

(i) Using Pythagoras

$$x^2 + h^2 = 4^2$$

$$x^2 = 16 - h^2$$
$$x = \sqrt{16 - h^2}$$

(ii) Find $\frac{dx}{dt}$ when $h = 1$

$$\frac{dx}{dt} = \frac{dx}{dh} \cdot \frac{dh}{dt}$$
$$x = \left(16 - h^2\right)^{\frac{1}{2}}$$
$$\frac{dx}{dh} = \frac{1}{2} \times \left(16 - h^2\right)^{-\frac{1}{2}} \times \frac{d}{dh}\left(16 - h^2\right)$$
$$= \frac{1}{2}\left(16 - h^2\right)^{-\frac{1}{2}} \times -2h$$
$$= \frac{-h}{\sqrt{16 - h^2}}$$

When $h = 1, \frac{dh}{dt} = -0.3$ m/hr

(h decreases when the tide is rising)

$$\frac{dx}{dt} = \frac{-h}{\sqrt{16 - h^2}} \times -0.3$$
$$= \frac{-1}{\sqrt{16 - 1^2}} \times -0.3$$
$$= \frac{0.3}{\sqrt{15}}$$
$$= 0.0774\ldots$$
$$= 0.077 \quad \text{metres per hr (to 2 d.p.)}$$

$\therefore$ When $h = 1$, the pontoon is moving away at 0.077 metres per hr.

---

5. Calculus in the Physical World, EXT1 2010 HSC 2d

Using Pythagoras

$$r^2 = x^2 + 6^2$$
$$r = \sqrt{x^2 + 36}, \ r > 0$$
$$\frac{dr}{dt} = \frac{dx}{dt} \cdot \frac{dr}{dx} \quad \text{... (1)}$$
$$\frac{dx}{dt} = 100 \quad \text{(given)}$$
$$\frac{dr}{dx} = \frac{1}{2}\left(x^2 + 36\right)^{-\frac{1}{2}} \times \frac{d}{dx}\left(x^2 + 36\right)$$
$$= \frac{x}{\sqrt{x^2 + 36}}$$

Substituting into (1)

$$\therefore \frac{dr}{dt} = \frac{100x}{\sqrt{x^2 + 36}} \text{ km/hr}$$

---

6. Calculus in the Physical World, EXT1 2006 HSC 5c

(i) Show $\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}$

$$\frac{dV}{dt} = k$$
$$V = \frac{\pi}{3}x^2(3r - x)$$
$$= r\pi x^2 - \frac{\pi}{3}x^3$$
$$\frac{dV}{dx} = 2\pi rx - \pi x^2$$
$$= \pi x(2r - x)$$

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$k = \pi x(2r - x) \cdot \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{k}{\pi x(2r - x)} \quad \ldots \text{ as required}$$

(ii) $$\frac{dx}{dt} = \frac{k}{\pi x(2r - x)}$$

$$\frac{dt}{dx} = \frac{1}{k}\pi x(2r - x)$$

$$t = \frac{1}{k}\int 2\pi rx - \pi x^2 \, dx$$

$$= \frac{1}{k}\left[\pi r x^2 - \frac{1}{3}\pi x^3\right] + c$$

When $t = 0, \quad x = 0$

$$\therefore c = 0$$

$$\therefore t = \frac{1}{k}\left[\pi r x^2 - \frac{1}{3}\pi x^3\right]$$

Find t_1, when $x = \frac{1}{3}r$

$$t_1 = \frac{1}{k}\left[\pi r\left(\frac{r}{3}\right)^2 - \frac{1}{3}\pi\left(\frac{r}{3}\right)^3\right]$$

$$= \frac{1}{k}\left[\frac{\pi r^3}{9} - \frac{\pi r^3}{81}\right]$$

$$= \frac{1}{k}\left(\frac{9\pi r^3}{81} - \frac{\pi r^3}{81}\right)$$

$$= \frac{8\pi r^3}{81k}$$

Find t_2 when $x = \frac{2}{3}r$

$$t_2 = \frac{1}{k}\left[\pi r\left(\frac{2r}{3}\right)^2 - \frac{1}{3}\pi\left(\frac{2r}{3}\right)^3\right]$$

$$= \frac{1}{k}\left[\frac{4\pi r^3}{9} - \frac{8\pi r^3}{81}\right]$$

$$= \frac{1}{k}\left(\frac{36\pi r^3}{81} - \frac{8\pi r^3}{81}\right)$$

$$= \frac{28\pi r^3}{81k}$$

$$= 3.5 \times \frac{8\pi r^3}{81k}$$

$$= 3.5 \times t_1$$

$\therefore$ It takes 3.5 times longer to fill the bowl.

---

7. Calculus in the Physical World, EXT1 2013 HSC 13a

(i) Need to show $\frac{dr}{dt}$ is a constant

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad \ldots (1)$$

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = -10^{-4}A \quad \text{(given)}$$

Substituting into (1)

$$-10^{-4}A = 4\pi r^2 \times \frac{dr}{dt}$$

$$= A \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = -10^{-4} \quad \ldots \text{ as required}$$

(ii) Using $V = 10^{-6}$ m³

$$\frac{4}{3}\pi r^3 = 10^{-6}$$

$$r^3 = \frac{3 \times 10^{-6}}{4\pi}$$

$$r = \sqrt[3]{\frac{3 \times 10^{-6}}{4\pi}}$$

Since the radius decreases at a constant rate,

♦♦ *Mean mark 31%*

$$t = \frac{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}{10^{-4}}$$

$= 62.035...$

$= 62$ seconds (nearest whole)

$\therefore$ It takes 62 seconds for then raindrop to evaporate.

---

8. Calculus in the Physical World, EXT1 2004 HSC 5a

(i) Show $\dot{x} = x^2 + 1$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$$

$$\therefore \frac{1}{2}v^2 = \int 2x^3 + 2x\, dx$$

$$= \frac{2}{4}x^4 + x^2 + c$$

$$v^2 = x^4 + 2x^2 + c$$

When $x = 2,\ v = 5$

$$5^2 = 2^4 + \left(2 \times 2^2\right) + c$$

$$25 = 16 + 8 + c$$

$$c = 1$$

$$\therefore v^2 = x^4 + 2x^2 + 1$$

$$= \left(x^2 + 1\right)^2$$

$$v = \sqrt{\left(x^2 + 1\right)^2}$$

$\therefore \dot{x} = x^2 + 1$ ... as required

(ii) $$\frac{dx}{dt} = x^2 + 1$$

$$\frac{dt}{dx} = \frac{1}{x^2 + 1}$$

$$\therefore t = \int \frac{1}{x^2 + 1}\, dx$$

$$= \tan^{-1} x + c$$

When $t = 0,\ x = 2$

$$0 = \tan^{-1} 2 + c$$

$$c = -\tan^{-1} 2$$

$$\therefore t = \tan^{-1} x - \tan^{-1} 2$$

$$\tan^{-1} x = t + \tan^{-1} 2$$

$$\therefore x = \tan\left(t + \tan^{-1} 2\right)$$

---

9. Calculus in the Physical World, EXT1 2011 HSC 7a

(i) Show that $V = \frac{\pi}{3}\left(h^3 - l^3\right)$

♦ *Mean mark 42%*

Since $\tan 45° = \frac{r}{h} = 1$

$\Rightarrow r = h$

$\Rightarrow$ Radius of lower cone $= h$

$\therefore$ $V(\text{lower cone}) = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi h^3$$

Similarly

$$V(\text{submerged upper cone}) = \frac{1}{3}\pi l^3$$

$$V(\text{water left}) = \frac{1}{3}\pi h^3 - \frac{1}{3}\pi l^3$$

$$= \frac{\pi}{3}\left(h^3 - l^3\right) \quad \text{... as required}$$

(ii) Find $\frac{dV}{dt}$ at $l = 2$

$$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt} \quad \text{... (1)}$$

$$\Rightarrow \frac{dl}{dt} = 10 \text{ (given)}$$

Using $V = \frac{\pi}{3}\left(h^3 - l^3\right)$ from part (i)

$$\Rightarrow \frac{dV}{dl} = -3 \times \frac{\pi}{3} l^2$$

$$= -\pi l^2$$

At $l = 2$

Substitute into (1) above

$$\frac{dV}{dt} = -\pi \times 2^2 \times 10$$

$$= -40\pi \quad \text{cm}^3/\text{sec}$$

(iii) Find $\frac{dV}{dt}$ when lower cone has lost $\frac{1}{8}$

Find l when $V = \frac{7}{8} \times \frac{1}{3}\pi h^3$

$$\frac{\pi}{3}\left(h^3 - l^3\right) = \frac{7}{8} \times \frac{1}{3}\pi h^3$$

$$h^3 - l^3 = \frac{7}{8}h^3$$

$♦♦♦$ *Mean mark 12%*

MARKER'S COMMENT: Many unsuccessful answers attempted to find an alternate version of $\frac{dV}{dt}$. Part (ii) directed students directly toward the correct strategy.

$$l^3 = \frac{1}{8}h^3$$

$$l = \frac{h}{2}$$

When $l = \frac{h}{2}$

$$\frac{dV}{dt} = -\pi\left(\frac{h}{2}\right)^2 \times 10 \quad \text{... (*)}$$

$$= \frac{-5\pi h^2}{2} \text{ cm}^3/\text{sec}$$

$\therefore V$ is decreasing at the rate of $\frac{5\pi h^2}{2}$ cm³/sec.